Information-Based Asset Pricing

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Abstract. A new framework for asset price dynamics is introduced in which the concept of noisy information about future cash flows is used to derive the corresponding price processes. In this framework an asset is defined by its cash-flow structure. Each cash flow is modelled by a random variable that can be expressed as a function of a collection of independent random variables called market factors. With each such market factor or "X-factor" we associate a so-called market information process, the values of which we assume are accessible to market participants. Each market information process consists of a sum of two terms; one contains true information about the value of the associated market factor, and the other represents "noise". The noise term is modelled by an independent Brownian bridge process that spans the time interval from the present to the time at which the value of the given market factor is revealed. The market filtration is assumed to be that generated by the aggregate of the independent market information processes. The price of an asset is given by the expectation of the discounted cash flows in the risk neutral measure, conditional on the information provided by the market filtration thus constructed. In the case where the cash flows are the random dividend payments associated with equities, an explicit model is obtained for the share-price process. Dividend growth is taken into account by introducing appropriate structure on the market factors. The prices of options on dividend-paying assets are derived. Remarkably, the resulting formula for the price of a European-style call option is of the Black-Scholes type. We consider both the case where the rate at which information is revealed to the market is constant, as well as the case where the information flow rate varies in time. Option pricing formulae are obtained for both cases. The information-based framework has another significant consequence: it generates a natural explanation for the origin of unhedgeable stochastic volatility in financial markets, without the need for specifying on an *ad hoc* basis the stochastic dynamics of the volatility.

Key words: Asset pricing; market information processes; stochastic volatility; correlation; dividend growth; X-factors; Brownian bridge; nonlinear filtering

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I. INTRODUCTION

In derivative pricing, the starting point is usually the specification of a model for the price process of the underlying asset. Such models generally tend to be of an *ad hoc* nature. For example, in the Black-Scholes theory, the underlying asset has a geometric Brownian motion as its price process. More generally, but equally arbitrarily, the economy is often modelled by a probability space equipped with the filtration generated by a multi-dimensional Brownian motion, and it is assumed that asset prices are Ito processes that are adapted to this filtration. This particular example is of course the "standard" model within which a great deal of financial engineering has been carried out.

The basic methodological problem with the standard model (and the same applies to various generalisations thereof) is that the market filtration is fixed once and for all, and little or no comment is offered on the issue of "where it comes from". In other words, the filtration, which represents the unfolding of information available to market participants, is modelled first, in an *ad hoc* manner, and then it is assumed that the asset price processes are adapted to it. But no indication is given about the nature of this "information", and it is not at all obvious, *a priori*, why the Brownian filtration, for example, should be regarded as providing information rather than simply noise.

To be sure, in a complete market there is a certain sense in which the Brownian filtration provides all of the relevant information, and no irrelevant information. That is to say, in a complete market based on a Brownian filtration the asset price movements precisely reflect the information content of the filtration. Nevertheless, the notion that the market filtration should in any simplistic sense be "prespecified" is an unsatisfactory one in financial modelling. The usual intuition behind the "prespecified-filtration" approach is to imagine that the filtration represents the unfolding in time of a succession of random events that "influence" the markets, thus causing prices to change. For example, a spell of bad weather in South America results in a decrease in the supply of coffee beans and hence an increase in the price of coffee. Or, say, a spate of bad derivative deals causes a drop in client confidence in investment banks, and hence a downgrade in earnings projections, and thus a drop in the share prices of these firms. The idea is that one then "abstractifies" these various influences in the form of a prespecified background filtration to which asset price processes are assumed to be adapted. What is unsatisfactory about this is that so little structure is given to the filtration: price movements behave as though they were spontaneous. In reality, we expect the price-formation process to exhibit more structure. It would be out of place, in the present context, to attempt anything like a complete account of the process of price formation. Nevertheless, we can try to improve on the "prespecified" approach. In that spirit we proceed as follows. We note that price changes arise from two rather distinct sources. The first source of price change is that resulting from changes in market-agent preferences—that is to say, changes in the pricing kernel. Movements in the pricing kernel are associated with (a) changes in investor attitudes towards risk, and (b) changes in investor "impatience", i.e. the subjective discounting of future cash flows. But equally important, if not more so, are those changes in price resulting from the revelation to market agents of information about the future cash flows derivable from possession of a given asset.

When a market agent decides to buy or sell an asset, the decision is made in accordance with the information available to the agent concerning the likely future cash flows associated with the asset. A change in the information available to the market agent about a future cash flow will typically have an effect on the price at which they are willing to buy or sell, even if the agent's preferences remain unchanged. Consider the situation where one is thinking of purchasing an item at a price that seems attractive. But then, by chance, one reads a newspaper article pointing out some undesirable feature of the product. After some reflection, one decides that the price is not so attractive, and in fact that the item is somewhat overpriced, considering the deficiencies that one is now aware of. As a result, one decides not to buy, not at that price, and eventually—possibly because many other individuals also have read the same report—the price drops.

The movement of the price of an asset should, therefore, be regarded as *an emergent phenomenon*. To put the matter another way, the price process of an asset should be viewed as the output of (rather than an input into) the various decisions made relating to possible transactions in the asset, and these decisions in turn should be understood as being induced primarily by the flow of information to market participants.

Taking into account this elementary observation we propose in this paper the outlines of a new framework for asset pricing based on *modelling of the flow of market information*. The information, more specifically, is that concerning the values of the future cash flows associated with the given assets. For example, if the asset represents a share in a firm that will make a single distribution at some pre-agreed date, then there is a single cash flow corresponding to the random amount of the distribution. If the asset is a credit-risky discount bond, then the future cash flow is the payout of the bond at the maturity date. In each case, based on the information available relating to the likely payouts of the given financial instrument, market participants determine, as best as they can, estimates for the value of the right to the impending cash flows. These estimates, in turn, lead to decisions concerning transactions, which then trigger movements in the price.

In this paper we present a simple class of models capturing the essence of the scenario described above. In building the framework described in what follows we have several criteria in mind that we would like to see satisfied. The first of these is that our model for the flow of market information should be intuitively appealing, and should allow for a reasonably sophisticated account of aggregate investor behaviour. At the same time, the model should be simple enough to allow one to derive explicit expressions for the asset price processes thus induced, in a suitably rich range of examples, as well as for various associated derivative price processes. The framework should also be flexible enough to allow for the modelling of assets having complex cash-flow structures. Furthermore, it should be suitable for practical implementation, with the property that calibration and pricing can be carried out swiftly and robustly, at least for more elementary structures. We would like the framework to be mathematically sound, and to be manifestly arbitrage-free. In what follows we shall show how our modelling framework goes a long way towards satisfying these diverse criteria.

The role of information in financial modelling has long been appreciated, particularly in the theory of market microstructure (see, e.g., Back [1], Back and Baruch [2], and references cited therein). The present framework is perhaps most closely related to the line of investigation represented, e.g., in Cetin, *et al.* [5], Duffie and Lando [9], Giesecke [10], Giesecke and Goldberg [11], Guo, *et al.* [13], and Jarrow and Protter [14]. The work in this paper, in particular, extends that described in Brody, *et al.* [3] (see also Rutkowski and Yu [20]).

The paper is organised as follows. In Section II we illustrate the basic framework for information-based pricing by considering the scenario in which there is a single random cash flow occurring at a designated time in the future. An elementary model for market information is presented, based on the specification of a process composed of two parts: a "signal" component containing true information about the upcoming cash flow, and an independent "noise" component which we model in a specific way. A closed-form expression for the asset price is obtained in terms of the market information available at the time the price is being specified. This result is summarised in Proposition 1. In Section III we show that the resulting asset price process is driven by a Brownian motion, an expression for which can be obtained in terms of the market information process: this construction indicates in explicit terms the sense in which the price process can be viewed as an "emergent" phenomenon. In Section IV we show that the value of a European-style call option, in the case of an asset with a single cash flow, admits a simple formula analogous to that of the Black-Scholes model. In Section V we derive pricing formulae for the situation when the random variable associated with the single cash flow has an exponential distribution or, more generally, a gamma distribution.

The extension of the framework to assets associated with multiple cash flows is established in Section VI. We show, in particular, that once the relevant cash flows are decomposed in terms of a collection of independent market factors, then a closed-form expression for the asset price associated with a complex cash-flow structure can be obtained. Moreover, by allowing distinct assets to share one or more common market factors in the determination of one or more of their respective cash flows, we obtain a natural correlation structure for the associated asset price processes. This method for introducing correlation in asset price movements contrasts sharply with the *ad hoc* approach adopted in most financial modelling. In Section VII we demonstrate that if two or more market factors affect the future cash flows associated with an asset, then the corresponding price process will exhibit unhedgeable stochastic volatility. This result is noteworthy because even for the class of relatively simple models considered here it is possible to identify a plausible candidate for *the origin of stochasticity in price volatility*, as well as the specific form it should take, which is given in Proposition 2.

In the remaining sections of the paper we generalise the previous discussion to the case where the rate at which the information concerning the true value of an impending cash flow is revealed is time dependent. The introduction of a time-dependent information flow rate adds additional flexibility to the modelling framework, and opens the door to the possibility of calibrating the resulting models to the market prices of families of options. We consider the single-factor case first, and obtain a closed-form expression for the conditional expectation of the cash flow. The result is stated first in Section VIII as Proposition 3, and the derivation is then given in the two sections that follow. Specifically in Section IX we introduce a new measure appropriate for the consideration of a Brownian bridge process with a random drift, which is used in Section X to obtain an expression for the conditional probability density function of the random cash flow. The dynamical consistency of the resulting asset price process is established in Section XI. We show, in particular, that, for the given information process, if we re-initialise the model at some specified future time, the dynamics of the model moving forward from that time can be represented by a suitably re-initialised information process. The precise statement of this result is given in Proposition 4.

The dynamical equation satisfied by the price process is analysed in Section XII, where we demonstrate in Proposition 5 that the driving process is a Brownian motion, just as in the constant parameter case. In Section XIII we derive the pricing formula for a European-style call option in the case for which the information flow rate is time dependent.

Our framework is based on the idea that first one models the cash flows, then the information processes, then the market filtration, and finally the price processes. In Section XIV, however, we solve the corresponding "inverse" problem. The result is stated in Proposition 6. Starting from the dynamics of the conditional probability distribution of the impending payoff, which is driven by a Brownian motion adapted to the market filtration, we construct (a) the random variable that represents the relevant market factor, and (b) an independent Brownian bridge process representing irrelevant information. These two then combine to generate the market filtration. We conclude in Section XV with a general multi-factor extension of the time-dependent setup, for which the dynamics of the resulting price processes are given in Propositions 7 and 8.

II. THE MODELLING FRAMEWORK

In asset pricing we require three basic ingredients, namely, (a) the cash flows, (b) the investor preferences, and (c) the flow of information available to market participants. Translated into somewhat more mathematical language, these ingredients amount to the following: (a′) cash flows are modelled as random variables; (b′) investor preferences are modelled with the determination of a pricing kernel; and (c′) the market information flow is modelled with the specification of a filtration. As we have indicated above, asset pricing theory conventionally attaches more weight to (a) and (b) than to (c). In this paper, however, we emphasise the importance of ingredient (c).

Our theory will be based on modelling the flow of information accessible to market participants concerning the future cash flows associated with the possession of an asset, or with a position in a financial contract. We start by setting the notation and introducing the assumptions employed in this paper. We model the financial markets with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on which a filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$ will be constructed. The probability measure $\mathbb{Q}$ is understood to be the risk-neutral measure, and the filtration $\{\mathcal{F}_t\}$ is understood to be the market filtration. All asset-price processes and other information-providing processes accessible to market participants will be adapted to $\{\mathcal{F}_t\}$. We do not regard $\{\mathcal{F}_t\}$ as something handed to us on a platter. Instead, it will be modelled explicitly. This will be undertaken shortly.

Several simplifying assumptions will be made. These assumptions should be regarded as being merely temporary, so that we can concentrate our efforts on the problems associated with the flow of market information. The first of these assumptions is the use of the risk-neutral measure. The "real" probability measure does not enter into the present investigation. We leap over that part of the economic analysis that determines the pricing measure. More specifically, we assume the absence of arbitrage and the existence of an established pricing kernel (see, e.g., Cochrane 2005, and references cited therein). With these conditions the existence of a unique risk-neutral pricing measure $\mathbb{Q}$ is ensured, even though the markets we consider will, in general, be incomplete.

Our second assumption is that we take the default-free system of interest rates to be deterministic. This is not to say that interest rate stochasticity should be ignored. Our view is rather that we should first develop our framework in a simplified setting, where certain essentially macroeconomic issues are put to one side; then, once we are satisfied with the tentative framework, we can attempt to generalise it in such a way as to address these issues. We therefore assume a deterministic default-free discount bond system. The absence of arbitrage implies that the corresponding system of discount functions $\{P_{tT}\}_{0\leq t\leq T<\infty}$ can be written in the form $P_{tT} = P_{0T}/P_{0t}$ for $t \leq T$, where $\{P_{0t}\}_{0\leq t<\infty}$ is the initial discount function, which we take to be part of the initial data of the model. The function $\{P_{0t}\}_{0\leq t<\infty}$ is assumed to be differentiable and strictly decreasing, and to satisfy $0 < P_{0t} \leq 1$ and $\lim_{t\to\infty} P_{0t} = 0$. These conditions can be relaxed somewhat for certain applications.

We also assume, for simplicity, that all cash flows occur at pre-determined dates. Now clearly for some purposes we would like to allow for cash flows occurring effectively at random times—in particular, at stopping times associated with the market filtration. But in the present exposition we want to avoid the idea of a "prespecified" filtration with respect to which stopping times are defined. We take the view that the market filtration is a "derived" notion, generated by information about impending cash flows, and by the actual values of cash flows when they occur. In the present paper we regard a "randomly-timed" cash flow as being a *set* of random cash flows occurring at various times—and with a joint distribution function that ensures only one of these flows is non-zero. Hence in our view the ontological status of a cash flow is that its timing is definite, only the amount is random—and that cash flows occurring at different times are, by their nature, different cash flows.

Modelling the cash flows. First we consider the case of a single isolated cash flow occurring at time T, represented by a random variable D_T. We assume that $D_T \geq 0$. The value S_t of the cash flow at any earlier time t in the interval $0 \leq t < T$ is then given by the discounted conditional expectation of D_T:

$$S_t = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[D_T|\mathcal{F}_t\right]. \tag{1}$$

In this way we model the price process $\{S_t\}_{0\leq t<T}$ of a limited-liability asset that pays the single dividend D_T at time T. The construction of the price process here is carried out in such a way as to guarantee an arbitrage-free market if other assets are priced by the same method (see Davis [7] for a closely related point of view). With a slight abuse of terminology we shall use the terms "cash flow" and "dividend" more or less interchangeably. If a more specific use of one of these terms is needed, then this will be evident from the context. We adopt the convention that when the dividend is paid the asset price goes "ex-dividend" immediately. Hence in the example above we have $\lim_{t\to T} S_t = D_T$ and $S_T = 0$.

In the case that the asset pays a sequence of dividends D_{T_k} $(k = 1, 2, \ldots, n)$ on the dates T_k the price (for values of t earlier than the time of the first dividend) is given by

$$S_t = \sum_{k=1}^{n} P_{tT_k}\mathbb{E}^{\mathbb{Q}}\left[D_{T_k}|\mathcal{F}_t\right]. \tag{2}$$

More generally, for all $t \geq 0$, and taking into account the ex-dividend behaviour, we have

$$S_t = \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} P_{tT_k}\mathbb{E}^{\mathbb{Q}}\left[D_{T_k}|\mathcal{F}_t\right]. \tag{3}$$

It turns out to be useful if we adopt the convention that a discount bond also goes ex-dividend on its maturity date. Thus in the case of a discount bond we assume that the price of the bond is given, for dates earlier than the maturity date, by the product of the principal and the relevant discount factor. But at maturity (when the principal is paid out) the value of the bond drops to zero. In the case of a coupon bond, there is likewise a downward jump in the price of the bond at the time a coupon is paid (the value lost may be captured back in the form of an "accrued interest" payment). In this way we obtain a consistent treatment of the "ex-dividend" behaviour of all of the asset price processes under consideration here. With this convention it follows, in particular, that all price processes have the property that they are right continuous with left limits.

Modelling the information flow. Now we present a simple model for the flow of market information. We consider first the case of a single distribution, occurring at time T, and assume that market participants have only partial information about the upcoming cash flow D_T. The information available in the market about the cash flow is assumed to be contained in a process $\{\xi_t\}_{0\leq t\leq T}$ defined by:

$$\xi_t = \sigma D_T t + \beta_{tT}. \tag{4}$$

We call $\{\xi_t\}$ the *market information process*. The information process is composed of two parts. The term $\sigma D_T t$ contains the "true information" about the upcoming dividend. This term grows in magnitude as t increases. The process $\{\beta_{tT}\}_{0\leq t\leq T}$ is a standard Brownian bridge over the time interval $[0, T]$. Thus $\beta_{0T} = 0$, $\beta_{TT} = 0$, and at time t the random variable β_{tT} has mean zero and variance $t(T-t)/T$; the covariance of β_{sT} and β_{tT} for $s \leq t$ is $s(T-t)/T$. We assume that D_T and $\{\beta_{tT}\}$ are independent. Thus the information contained in the bridge process is "pure noise". The information contained in $\{\xi_t\}$ is clearly unchanged if we multiply $\{\xi_t\}$ by some overall scale factor.

We assume that the market filtration $\{\mathcal{F}_t\}$ is generated by the market information process. That is to say, we assume that $\{\mathcal{F}_t\} = \{\mathcal{F}^{\xi}_t\}$, where $\{\mathcal{F}^{\xi}_t\}$ is the filtration generated by $\{\xi_t\}$. The dividend D_T is therefore $\mathcal{F}_T$-measurable, but is not $\mathcal{F}_t$-measurable for $t < T$. Thus the value of D_T becomes "known" at time T, but not earlier. The bridge process $\{\beta_{tT}\}$ is not adapted to $\{\mathcal{F}_t\}$ and thus is not directly accessible to market participants. This reflects the fact that until the dividend is paid the market participants cannot distinguish the "true information" from the "noise" in the market.

The introduction of the Brownian bridge models the fact that market perceptions, whether valid or not, play a role in determining asset prices. Initially, all available information is used to determine the *a priori* risk-neutral probability distribution for D_T. Then after the passage of time rumours, speculations, and general disinformation start circulating, reflected in the steady increase in the variance of the Brownian bridge. Eventually the variance drops and falls to zero at the time the distribution to the share-holders is made. The parameter σ represents the rate at which information about the true value of D_T is revealed as time progresses. If σ is low, the value of D_T is effectively hidden until very near the time of the dividend payment; whereas if σ is high, then the value of the cash flow is for all practical purposes revealed very quickly.

In the example under consideration we have made some simplifying assumptions concerning our choice for the market information structure. For instance, we assume that σ is constant. In Section XII we consider a time-dependent market information flow rate. We have also assumed that the random dividend D_T enters directly into the structure of the information process, and enters linearly. As we shall indicate later, a more general and in some respects more natural setup is to let the information process depend on a random variable X_T which we call a "market factor"; then the dividend is regarded as a function of the market factor. This arrangement has the advantage that it easily generalises to the situation where a cash flow might depend on several independent market factors, or indeed where cash flows associated with different financial instruments have one or more market factors in common. But for the moment we regard the single cash flow D_T as being the relevant market factor, and we assume the information-flow rate to be constant.

With the market information structure described above for a single cash flow in place, we proceed to construct the associated price dynamics. The price process $\{S_t\}$ for a share in the firm paying the specified dividend is given by formula (1). It is assumed that the *a priori* probability distribution of the dividend D_T is known. This distribution is regarded as part of the initial data of the problem, which in some cases can be calibrated from knowledge of the initial price of the asset along possibly with other price data. The general problem of how the *a priori* distribution is obtained is an important one—any asset pricing model has to confront some version of this issue—which we defer for later consideration. The main point is that the initial distribution is not to be understood as being "absolutely" determined, but rather represents the "best estimate" for the distribution given the data available at that time, in accordance with what one might call a Bayesian point of view. We note the fact that the information process $\{\xi_t\}$ is Markovian (see Brody, *et al.* [3], and Rutkowski and Yu [20]). Making use of this property of the information process together with the fact that D_T is $\mathcal{F}_T$-measurable we deduce that

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}} [D_T | \xi_t] . \tag{5}$$

If the random variable D_T that represents the payoff has a continuous distribution, then the conditional expectation in (5) can be expressed in the form

$$\mathbb{E}^{\mathbb{Q}} [D_T | \xi_t] = \int_0^\infty x \pi_t(x) \, \mathrm{d}x. \tag{6}$$

Here $\pi_t(x)$ is the conditional probability density for the random variable D_T:

$$\pi_t(x) = \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{Q}(D_T \leq x | \xi_t). \tag{7}$$

We implicitly assume throughout the paper appropriate technical conditions on the distribution of the dividend that will suffice to ensure the existence of the expressions under consideration. Also, for convenience we use a notation appropriate for continuous distributions, though corresponding results can easily be inferred for discrete distributions, or more general distributions, by slightly modifying the stated assumptions and conclusions.

Bearing in mind these points, we note that the conditional probability density process for the dividend can be worked out explicitly by use of a form of the Bayes formula:

$$\pi_t(x) = \frac{p(x)\rho(\xi_t | D_T = x)}{\int_0^\infty p(x)\rho(\xi_t | D_T = x)\mathrm{d}x}. \tag{8}$$

Here $p(x)$ denotes the *a priori* probability density function for D_T, which we assume is known as an initial condition, and $\rho(\xi_t | D_T = x)$ denotes the conditional density function for the random variable ξ_t given that $D_T = x$. Since β_{tT} is a Gaussian random variable with variance $t(T-t)/T$, we deduce that the conditional probability density for ξ_t is

$$\rho(\xi_t | D_T = x) = \sqrt{\frac{T}{2\pi t (T-t)}} \exp\left(-\frac{(\xi_t - \sigma t x)^2 T}{2t(T-t)} \right). \tag{9}$$

Inserting the expression into the Bayes formula we get

$$\pi_t(x) = \frac{p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right]}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}. \tag{10}$$

We thus obtain the following result for the asset price:

Proposition 1. *The information-based price process $\{S_t\}_{0\leq t\leq T}$ of a limited-liability asset that pays a single dividend D_T at time T with a priori distribution*

$$\mathbb{Q}(D_T \leq y) = \int_0^y p(x) \, \mathrm{d}x \tag{11}$$

is given by

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \frac{\int_0^\infty x p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}, \tag{12}$$

where $\xi_t = \sigma D_T t + \beta_{tT}$ is the market information.

III. ASSET PRICE DYNAMICS IN THE CASE OF A SINGLE CASH FLOW

In order to analyse the properties of the price process deduced above, and to be able to compare it with other models, we need to work out the dynamics of $\{S_t\}$. One of the advantages of the model under consideration is that we have a completely explicit expression for the price process at our disposal. Thus in obtaining the dynamics we need to find the stochastic differential equation of which $\{S_t\}$ is the solution. This turns out to be an interesting exercise because it offers some insights into what we mean by the assertion that market price dynamics should be regarded as constituting an "emergent phenomenon".

To obtain the dynamics associated with the price process $\{S_t\}$ of a single-dividend paying asset let us write

$$D_{tT} = \mathbb{E}^{\mathbb{Q}}[D_T|\xi_t] \tag{13}$$

for the conditional expectation of D_T with respect to the market information ξ_t. Evidently, D_{tT} can be expressed in the form $D_{tT} = D(\xi_t, t)$, where the function $D(\xi, t)$ is defined by

$$D(\xi, t) = \frac{\int_0^\infty x p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}. \tag{14}$$

A straightforward calculation making use of the Ito rules shows that the dynamical equation for the conditional expectation $\{D_{tT}\}$ is given by

$$\mathrm{d}D_{tT} = \frac{\sigma T}{T-t} V_t \left[\frac{1}{T-t}\Big(\xi_t - \sigma T D_{tT}\Big)\mathrm{d}t + \mathrm{d}\xi_t\right]. \tag{15}$$

Here V_t is the conditional variance of the dividend:

$$V_t = \int_0^\infty x^2 \pi_t(x)\, \mathrm{d}x - \left(\int_0^\infty x \pi_t(x)\, \mathrm{d}x\right)^2. \tag{16}$$

Therefore, if we define a new process $\{W_t\}_{0\leq t<T}$ by setting

$$W_t = \xi_t - \int_0^t \frac{1}{T-s}\Big(\sigma T D_{tT} - \xi_s\Big)\mathrm{d}s, \tag{17}$$

we find, after some rearrangement of terms, that

$$\mathrm{d}D_{tT} = \frac{\sigma T}{T-t} V_t \mathrm{d}W_t. \tag{18}$$

For the dynamics of the asset price process we thus have

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \Gamma_{tT}\mathrm{d}W_t, \tag{19}$$

where the short rate r_t is given by $r_t = -\mathrm{d}\ln P_{0t}/\mathrm{d}t$, and the absolute price volatility Γ_{tT} is

$$\Gamma_{tT} = P_{tT}\frac{\sigma T}{T-t} V_t. \tag{20}$$

A slightly different way of arriving at this result is as follows. We start with the conditional probability process $\pi_t(x)$. Then, using the same notation as above, for the dynamics of $\pi_t(x)$ we obtain

$$\mathrm{d}\pi_t(x) = \frac{\sigma T}{T-t}(x - D_{tT})\pi_t(x)\,\mathrm{d}W_t. \tag{21}$$

Since the asset price is given by

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \int_0^\infty x\pi_t(x)\,\mathrm{d}x, \tag{22}$$

we are thus able to infer the dynamics of the price $\{S_t\}$ from the dynamics of the conditional probability $\{\pi_t(x)\}$, once we take into account the formula for the conditional variance.

As we shall demonstrate later, the process $\{W_t\}$ defined in (17) is an $\{\mathcal{F}_t\}$-Brownian motion. Hence *from the point of view of the market it is the process* $\{W_t\}$ *that drives the asset price dynamics*. In this way our framework resolves the somewhat paradoxical point of view usually adopted in financial modelling in which $\{W_t\}$ is regarded as "noise", and yet also generates the market information flow. And thus, instead of simply hypothesising the existence of a driving process for the dynamics of the markets, we are able from the information-based perspective to *deduce* the existence of such a process.

The information-flow parameter σ determines the overall magnitude of the volatility. In fact, the parameter σ plays a role that is in many respects analogous to the similarly-labelled parameter in the Black-Scholes theory. Thus, we can say that the rate at which information is revealed in the market determines the overall magnitude of the market volatility. In other words, everything else being the same, if we increase the information flow rate, then the market volatility will increase as well. It is ironic that, according to this point of view, those mechanisms that one might have thought were destined to make markets more efficient—e.g., globalisation of the financial markets, reduction of trade barriers, improved communications, a more robust regulatory environment, and so on—can have the effect of increasing market volatility, and hence market risk, rather than reducing it.

IV. EUROPEAN-STYLE CALL OPTIONS

Before we turn to the consideration of more general cash flows and more general market information structures, let us consider the problem of pricing a derivative on an asset for which the price process is governed by the dynamics (19). It turns out that a complete treatment of this problem can be given. Specifically, we consider the valuation problem for a European-style call option on such an asset, with strike price K, and exercisable at a fixed maturity date t. The option is written on an asset that pays a single dividend D_T at time $T > t$. The value of the option at time 0 is clearly

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(S_t - K)^+\right]. \tag{23}$$

Inserting the information-based expression for the price S_t derived in the previous section into this formula, we obtain

$$C_0 = P_{0t}\,\mathbb{E}^{\mathbb{Q}}\left[\left(P_{tT}\int_0^\infty x\,\pi_t(x)\mathrm{d}x - K\right)^+\right]. \tag{24}$$

For convenience we write the conditional probability $\pi_t(x)$ in the form

$$\pi_t(x) = \frac{p_t(x)}{\int_0^\infty p_t(x)\mathrm{d}x}, \tag{25}$$

where the "unnormalised" density process $\{p_t(x)\}$ is defined by

$$p_t(x) = p(x) \exp\left[\frac{T}{T-t}\left(\sigma x \xi_t - \tfrac{1}{2}\sigma^2 x^2 t\right)\right]. \tag{26}$$

Substituting (26) into (24) we find that the initial value of the option is given by

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Lambda_t}\left(\int_0^\infty (P_{tT}x - K)\, p_t(x)\mathrm{d}x\right)^+\right], \tag{27}$$

where

$$\Lambda_t = \int_0^\infty p_t(x)\mathrm{d}x. \tag{28}$$

The random variable Λ_t can be used to introduce a measure $\mathbb{B}_T$ applicable over the time horizon $[0,t]$, which we call the "bridge measure". The call option price can thus be written:

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left(\int_0^\infty (P_{tT}x - K)\, p_t(x)\mathrm{d}x\right)^+\right]. \tag{29}$$

The special feature of the bridge measure, as we shall establish in Section IX in a somewhat more general context, is that the random variable ξ_t is Gaussian under $\mathbb{B}_T$. In particular, under the measure $\mathbb{B}_T$ we find that $\{\xi_t\}$ has mean 0 and variance $t(T-t)/T$. Since $p_t(x)$ can be expressed as a function of ξ_t, when we carry out the expectation above we are led to a tractable formula for C_0.

To obtain the value of the option we define a constant ξ^* (the critical value) by the following condition:

$$\int_0^\infty (P_{tT}x - K)\, p(x) \exp\left[\frac{T}{T-t}\left(\sigma x \xi^* - \tfrac{1}{2}\sigma^2 x^2 t\right)\right]\mathrm{d}x = 0. \tag{30}$$

Then the option price is given by:

$$C_0 = P_{0T}\int_0^\infty x\, p(x)\, N\Big(-z^* + \sigma x\sqrt{\tau}\Big)\mathrm{d}x - P_{0t}K\int_0^\infty p(x)\, N\Big(-z^* + \sigma x\sqrt{\tau}\Big)\mathrm{d}x, \tag{31}$$

where

$$\tau = \frac{tT}{T-t}, \qquad z^* = \xi^*\sqrt{\frac{T}{t(T-t)}}, \tag{32}$$

and $N(x)$ denotes the standard normal distribution function. We see that a tractable expression is obtained, and that it is of the Black-Scholes type. The option pricing problem, even for general $p(x)$, reduces to an elementary numerical problem. It is interesting to note that although the probability distribution for the price S_t at time t is not of a "standard" type, nevertheless the option valuation problem remains a solvable one.

V. EXAMPLES OF SPECIFIC DIVIDEND STRUCTURES

In this section we consider the dynamics of assets with various specific dividend structures. First we look at a simple asset for which the cash flow is exponentially distributed. The *a priori* probability density for D_T is thus of the form

$$p(x) = \frac{1}{\delta}\,\exp\left(-x/\delta\right), \tag{33}$$

where δ is a constant. The idea of an exponentially distributed payout is of course somewhat artificial. Nevertheless we can regard this as a useful model for the situation where little is known about the probability distribution of the dividend, apart from its mean. Then from formula (12) we find that the corresponding asset price is given by:

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \frac{\int_0^\infty x \exp(-x/\delta) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}{\int_0^\infty \exp(-x/\delta) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}. \tag{34}$$

We note that $S_0 = P_{0T}\delta$, so we can calibrate the choice of δ by use of the initial price. The integrals in the numerator and denominator in the expression above can be worked out explicitly. Hence, we obtain a closed-form expression for the asset price in the case of a simple asset with an exponentially-distributed terminal cash flow. This is given by:

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \left[\frac{\exp\left(-\frac{1}{2}B_t^2/A_t\right)}{\sqrt{2\pi A_t}\; N(B_t/\sqrt{A_t})} + \frac{B_t}{A_t} \right], \tag{35}$$

where $A_t = \sigma^2 tT/(T-t)$ and $B_t = \sigma T \xi_t/(T-t) - \delta^{-1}$.

Next we consider the case of an asset for which the single dividend paid at time T is gamma-distributed. More specifically, we assume the probability density is of the form

$$p(x) = \frac{\delta^n}{(n-1)!} x^{n-1} \exp(-\delta x), \tag{36}$$

where δ is a positive real number and n is a positive integer. This choice for the probability density also leads to a closed-form expression for the share price. We find that

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \frac{\sum_{k=0}^{n} \binom{n}{k} A_t^{\frac{1}{2}k-n} B_t^{n-k} F_k(-B_t/\sqrt{A_t})}{\sum_{k=0}^{n-1} \binom{n-1}{k} A_t^{\frac{1}{2}k-n+1} B_t^{n-k-1} F_k(-B_t/\sqrt{A_t})}, \tag{37}$$

where A_t and B_t are as above, and

$$F_k(x) = \int_x^\infty z^k \exp\left(-\tfrac{1}{2}z^2\right) dz. \tag{38}$$

A recursion formula can be worked out for the function $F_k(x)$. This is given by

$$(k+1)F_k(x) = F_{k+2}(x) - x^{k+1}\exp\left(-\tfrac{1}{2}x^2\right), \tag{39}$$

from which it follows that $F_0(x) = \sqrt{2\pi}N(-x)$, $F_1(x) = \mathrm{e}^{-\frac{1}{2}x^2}$, $F_2(x) = x\mathrm{e}^{-\frac{1}{2}x^2} + \sqrt{2\pi}N(-x)$, $F_3(x) = (x^2+2)\mathrm{e}^{-\frac{1}{2}x^2}$, and so on. In general, the polynomial parts of $\{F_k(x)\}_{k=0,1,2,\ldots}$ are related to the Legendre polynomials.

VI. MULTIPLE CASH FLOWS

In this section we generalise the preceding material to the situation where the asset pays multiple dividends. This will allow us to consider a wider range of financial instruments. Let us write D_{T_k} ($k = 1, \ldots, n$) for a set of random cash flows paid at the pre-designated dates T_k ($k = 1, \ldots, n$). Thus possession of the asset at time t entitles the bearer to the cash flows occurring at times $T_k > t$. For simplicity we assume n is finite, although with technical refinements the extension to infinite sequences of cash flows is also possible. For each value of k we introduce a set of independent random variables $X^{\alpha}_{T_k}$ ($\alpha = 1, \ldots, m_k$), which we call market factors or X-factors. For each value of α we assume that the market factor $X^{\alpha}_{T_k}$ is $\mathcal{F}_{T_k}$-measurable, where $\{\mathcal{F}_t\}$ is the market filtration.

Intuitively speaking, for each value of k the market factors $\{X^{\alpha}_{T_j}\}_{j\leq k}$ represent the independent elements that determine the cash flow occurring at time T_k. Thus for each value of k the cash flow D_{T_k} is assumed to have the following structure:

$$D_{T_k} = \Delta_{T_k}(X^{\alpha}_{T_1}, X^{\alpha}_{T_2}, ..., X^{\alpha}_{T_k}), \tag{40}$$

where $\Delta_{T_k}(X^{\alpha}_{T_1}, X^{\alpha}_{T_2}, ..., X^{\alpha}_{T_k})$ is a function of $\sum_{j=1}^{k} m_j$ variables. For each cash flow it is, so to speak, the job of the financial analyst (or actuary) to determine the relevant independent market factors, and the form of the cash-flow function Δ_{T_k} for each cash flow. With each market factor $X^{\alpha}_{T_k}$ we associate an information process $\{\xi^{\alpha}_{tT_k}\}_{0\leq t\leq T_k}$ of the form

$$\xi^{\alpha}_{tT_k} = \sigma^{\alpha}_{T_k} X^{\alpha}_{T_k} t + \beta^{\alpha}_{tT_k}. \tag{41}$$

Here $\sigma^{\alpha}_{T_k}$ is an information flux parameter, and $\{\beta^{\alpha}_{tT_k}\}$ is a standard Brownian bridge process over the interval $[0, T_k]$. We assume that the X-factors and the Brownian bridge processes are all independent of one another. The parameter $\sigma^{\alpha}_{T_k}$ determines the rate at which the true information about the value of the market factor $X^{\alpha}_{T_k}$ is revealed. The Brownian bridge $\beta^{\alpha}_{tT_k}$ represents the associated noise. We assume that the market filtration $\{\mathcal{F}_t\}$ is generated by the totality of the independent information processes $\{\xi^{\alpha}_{tT_k}\}_{0\leq t\leq T_k}$ for $k = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m_k$. Hence, the price process of the asset is given by

$$S_t = \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}}\left[D_{T_k} \middle| \mathcal{F}_t \right]. \tag{42}$$

Dividend growth. As an elementary example of a multi-dividend structure, we shall look at a simple *growth model* for dividends in the equity markets. We consider an asset that pays a sequence of dividends D_{T_k}, where each dividend date has an associated X-factor. Let $\{X_{T_k}\}_{k=1,\ldots,n}$ be a set of independent, identically-distributed X-factors, each with mean $1 + g$. The dividend structure is assumed to be of the form

$$D_{T_k} = D_0 \prod_{j=1}^{k} X_{T_j}, \tag{43}$$

where D_0 is a constant. The parameter g can be interpreted as the dividend growth factor, and D_0 can be understood as representing the most recent dividend before time zero. For the price process of the asset we have:

$$S_t = D_0 \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}}\left[\prod_{j=1}^{k} X_{T_j} \middle| \mathcal{F}_t \right]. \tag{44}$$

Since the X-factors are independent of one another, the conditional expectation of the product appearing in this expression factorises into a product of conditional expectations, and each such conditional expectation can be written in the form of an expression of the type we have already considered. As a consequence we are led to a completely tractable family of dividend growth models.

Assets with common factors. The multiple-dividend asset pricing model introduced in this section can be extended in a very natural way to the situation where two or more assets are being priced. In this case we consider a collection of N assets with price processes $\{S_t^{(i)}\}_{i=1,2,\ldots,N}$. With asset number (i) we associate the cash flows $\{D_{T_k}^{(i)}\}$ paid at the dates $\{T_k\}_{k=1,2,\ldots,n}$. We note that the dates $\{T_k\}_{k=1,2,\ldots,n}$ are not tied to any one specific asset, but rather represent the totality of possible cash-flow dates of any of the given assets. If a particular asset has no cash flow on one of the given dates, then it is simply assigned a zero cash-flow for that date. From this point, the theory proceeds exactly as in the single asset case. That is to say, with each value of k we associate a set of X-factors $X_{T_k}^{\alpha}$ $(\alpha = 1, 2, \ldots, m_k)$, and a corresponding system of market information processes, $\{\xi_{tT_k}^{\alpha}\}$. The X-factors and the information processes are not tied to any particular asset. The cash flow $D_{T_k}^{(i)}$ occurring at time T_k for asset number (i) is assumed to be given by a cash flow function of the form

$$D_{T_k}^{(i)} = \Delta_{T_k}^{(i)}(X_{T_1}^{\alpha}, X_{T_2}^{\alpha}, ..., X_{T_k}^{\alpha}). \tag{45}$$

In other words, for each asset each cash flow can depend on all of the X-factors that have been "activated" at that point. In particular, it is possible for two or more assets to "share" an X-factor in association with one or more of the cash flows of each of the assets. This in turn implies that the various assets will have at least one Brownian motion in common in the dynamics of their price processes. As a consequence we thus obtain a natural model for the existence of correlation structures in the prices of these assets. The intuition is that as new information comes in (whether "true" or "bogus") there will be several different assets all affected by the news, and as a consequence there will be a correlated movement in their prices. Thus for the general multi-asset model we have the following price process system:

$$S_t^{(i)} = \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[D_{T_k}^{(i)} | \mathcal{F}_t \right]. \tag{46}$$

VII. ORIGIN OF UNHEDGEABLE STOCHASTIC VOLATILITY

Based on the general model introduced in the previous section, we are now in a position to make an interesting observation concerning the nature of stochastic volatility in the equity markets. In particular, we shall show how *unhedgeable stochastic volatility* arises naturally in the information-based framework. This is achieved without the need for any *ad hoc* assumptions concerning the dynamics of the stochastic volatility. In fact, a very specific dynamical model for stochastic volatility is obtained—thus leading to a possible means by which the theory proposed here might be tested.

We shall work out the volatility associated with the dynamics of the general asset price process $\{S_t\}$ given by equation (42). The result is given in Proposition 2 below. First, as an example, we consider the dynamics of an asset that pays a single dividend D_T at time T. We assume that the dividend depends on a set of market factors $\{X_T^{\alpha}\}_{\alpha=1,\ldots,m}$. For $t < T$

we then have:

$$\begin{aligned} S_t &= P_{tT}\mathbb{E}^{\mathbb{Q}}\left[\Delta_T\left(X_T^1,\ldots,X_T^m\right)\middle|\,\xi_{tT}^1,\ldots,\xi_{tT}^m\right] \\ &= P_{tT}\int\cdots\int \Delta_T(x^1,\ldots,x^m)\,\pi_{tT}^1(x_1)\cdots\pi_{tT}^m(x_m)\,\mathrm{d}x_1\cdots\mathrm{d}x_m. \end{aligned} \tag{47}$$

Here the various conditional probability density functions $\pi_{tT}^{\alpha}(x)$ for $\alpha = 1,\ldots,m$ are

$$\pi_{tT}^{\alpha}(x) = \frac{p^{\alpha}(x)\exp\left[\frac{T}{T-t}\left(\sigma^{\alpha}\,x\,\xi_{tT}^{\alpha} - \frac{1}{2}(\sigma^{\alpha})^2\,x^2t\right)\right]}{\int_0^{\infty} p^{\alpha}(x)\exp\left[\frac{T}{T-t}\left(\sigma^{\alpha}\,x\,\xi_{tT}^{\alpha} - \frac{1}{2}(\sigma^{\alpha})^2\,x^2t\right)\right]\mathrm{d}x}, \tag{48}$$

where $p^{\alpha}(x)$ denotes the *a priori* probability density function for the market factor X_T^{α}. The drift of $\{S_t\}_{0\le t<T}$ is given by the short rate of interest. This is because $\mathbb{Q}$ is the risk-neutral measure, and no dividend is paid before T. Thus, we are left with the problem of determining the volatility of $\{S_t\}$. We find that for $t<T$ the dynamical equation of $\{S_t\}$ assumes the following form:

$$\mathrm{d}S_t = r_tS_t\mathrm{d}t + \sum_{\alpha=1}^{m}\Gamma_{tT}^{\alpha}\mathrm{d}W_t^{\alpha}. \tag{49}$$

Here the volatility term associated with factor number α is given by

$$\Gamma_{tT}^{\alpha} = \sigma^{\alpha}\frac{T}{T-t}P_{tT}\,\mathrm{Cov}\left[\Delta_T\left(X_T^1,\ldots,X_T^m\right),X_T^{\alpha}\middle|\,\mathcal{F}_t\right], \tag{50}$$

and $\{W_t^{\alpha}\}$ denotes the Brownian motion associated with the information process $\{\xi_t^{\alpha}\}$, as defined in (17). The absolute volatility of $\{S_t\}$ is evidently of the form

$$\Gamma_t = \left(\sum_{\alpha=1}^{m}\left(\Gamma_{tT}^{\alpha}\right)^2\right)^{1/2}. \tag{51}$$

For the dynamics of a multi-factor single-dividend paying asset we can thus write

$$\mathrm{d}S_t = r_tS_t\mathrm{d}t + \Gamma_t\mathrm{d}Z_t, \tag{52}$$

where the $\{\mathcal{F}_t\}$-Brownian motion $\{Z_t\}$ that drives the asset-price process is defined by

$$Z_t = \int_0^t \frac{1}{\Gamma_s}\sum_{\alpha=1}^{m}\Gamma_{sT}^{\alpha}\,\mathrm{d}W_s^{\alpha}. \tag{53}$$

The key point to note here is that in the case of a multi-factor model we obtain an unhedgeable stochastic volatility. That is to say, although the asset price is in effect driven by a single Brownian motion, its volatility in general depends on a multiplicity of Brownian motions. This means that in general an option position cannot be hedged with a position in the underlying asset. The components of the volatility vector are given by the covariances of the terminal cash flow and the independent market factors. Unhedgeable stochastic volatility thus emerges from the multiplicity of uncertain elements in the market that affect the value of the future cash flow. As a consequence we see that *in this framework we obtain a natural explanation for the origin of stochastic volatility in the equity markets.*

This result can be contrasted with, say, the Heston model [12], which despite its wide popularity suffers somewhat from the fact that it is essentially *ad hoc* in nature. Much the same has to be said for the various generalisations of the Heston model that have been so widely used in commercial applications. The approach to stochastic volatility proposed in the present paper is thus of a fundamentally new character.

Expression (49) generalises naturally to the case for which the asset pays a set of dividends D_{T_k} $(k = 1, \ldots, n)$, and for each k the dividend depends on the X-factors $\{\{X^{\alpha}_{T_j}\}^{\alpha=1,\ldots,m_j}_{j=1,\ldots,k}\}$. The result can be summarised as follows.

Proposition 2. *The price process of a multi-dividend asset has the following dynamics:*

$$\begin{aligned} \mathrm{d}S_t \;&=\; r_t\, S_t\, \mathrm{d}t + \sum_{k=1}^{n}\sum_{\alpha=1}^{m_k} \mathbf{1}_{\{t<T_k\}} \frac{\sigma^{\alpha}_{k} T_k}{T_k - t}\, P_{tT_k}\, \mathrm{Cov}\left[\Delta_{T_k}, X^{\alpha}_{T_k} \middle| \mathcal{F}_t\right] \mathrm{d}W^{\alpha k}_t \\ &\quad + \sum_{k=1}^{n} \Delta_{T_k} \mathrm{d}\mathbf{1}_{\{t<T_k\}}, \end{aligned} \tag{54}$$

where $\Delta_{T_k} = \Delta_{T_k}(X^{\alpha}_{T_1}, X^{\alpha}_{T_2}, \cdots, X^{\alpha}_{T_k})$ *is the dividend at time* T_k $(k = 1, 2, \ldots, n)$.

VIII. TIME-DEPENDENT INFORMATION FLUX

In the remainder of this paper we consider a generalisation of the foregoing material to the situation in which the information-flow rate varies in time. The time-dependent problem is of relevance to many circumstances. For example, there will typically be more activity in a market during the day than at night—such a consideration is important for short-term investments. Alternatively, it may be that the annual report of a firm is going to be published on a specified day—in this case much more information concerning the future of the firm may be made available on that day than normal.

We begin our analysis of the time-dependent case by considering the situation where there is a single cash flow D_T occurring at time T, and the associated market factor is taken to be the cash flow itself. In this way we can focus our attention on mathematical issues arising from the time dependence of the information flow rate. Once these issues have been dealt with, we shall consider more complicated cash-flow structures in Section XV. For the market information process we propose an expression of the form

$$\xi_t = D_T \int_0^t \sigma_s \mathrm{d}s + \beta_{tT}, \tag{55}$$

where the function $\{\sigma_s\}_{o\leq s\leq T}$ is taken to be deterministic and nonnegative. We assume that $0 < \int_0^T \sigma_s^2 ds < \infty$. The price process $\{S_t\}$ of the asset is given by

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}}\left[D_T \,| \mathcal{F}_t\right]. \tag{56}$$

where the market filtration is, as in the previous sections, assumed to be generated by the information process $\{\xi_t\}$, and $\mathbb{Q}$ denotes the risk-neutral measure.

Our first task is to work out the conditional expectation in (56). This can be achieved by use of a change-of-measure technique, which will be outlined in Section IX. It will be useful, however, to state the result first. We define the conditional probability density process $\{\pi_t(x)\}$ by setting

$$\pi_t(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,\mathbb{Q}\left(D_T \le x | \mathcal{F}_t\right). \tag{57}$$

The following result is obtained:

Proposition 3. *Let the information process $\{\xi_t\}$ be given by* (55)*. Then the conditional probability density process $\{\pi_t(x)\}$ for the random variable D_T is given by*

$$\pi_t(x) = \frac{p(x)\,\mathrm{e}^{x\left(\frac{1}{T-t}\,\xi_t\int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)}}{\int_0^\infty p(x)\,\mathrm{e}^{x\left(\frac{1}{T-t}\,\xi_t\int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)}\mathrm{d}x}. \tag{58}$$

We deduce at once from Proposition 3 that the conditional expectation of the random variable D_T is

$$D_{tT} = \frac{\int_0^\infty x p(x)\,\mathrm{e}^{x\left(\frac{1}{T-t}\,\xi_t\int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)}\mathrm{d}x}{\int_0^\infty p(x)\,\mathrm{e}^{x\left(\frac{1}{T-t}\,\xi_t\int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)}\mathrm{d}x}. \tag{59}$$

The associated price process $\{S_t\}$ is therefore given by $S_t = \mathbf{1}_{\{t<T\}} P_{tT} D_{tT}$.

IX. CHANGES OF MEASURE FOR BROWNIAN BRIDGES

Since the information process is a Brownian bridge with a random drift, we will require formulae relating a Brownian bridge with drift in one measure to a standard Brownian bridge in another measure to establish Proposition 3 . We proceed as follows. First we recall a well-known integral representation for the Brownian bridge. Let the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ be given, with a filtration $\{\mathcal{G}_t\}_{0\le t<\infty}$, and let $\{B_t\}$ be a standard $\{\mathcal{G}_t\}$-Brownian motion. Then the process $\{\beta_{tT}\}$, defined by

$$\beta_{tT} = (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}B_s, \tag{60}$$

for $0 \le t < T$, and by $\beta_{tT} = 0$ for $t = T$, is a standard Brownian bridge over the time interval $[0, T]$. The expression defined by (60) converges to zero as $t \to T$; see, e.g., Karatzas and Shreve [16], Protter [19]). The filtration $\{\mathcal{G}_t\}$ is larger than the market filtration $\{\mathcal{F}_t\}$. In particular, since $\{\beta_{tT}\}$ is adapted to $\{\mathcal{G}_t\}$ we can think of $\{\mathcal{G}_t\}$ as the filtration describing the information available to an omniscient "insider" who can distinguish between what is noise and what is not.

Let D_T be a random variable on $(\Omega, \mathcal{F}, \mathbb{Q})$. We assume that D_T is $\mathcal{G}_0$-measurable and that D_T is independent of $\{\beta_{tT}\}$. Thus the value of D_T is known "all along" to the omniscient insider, but not of course to the typical market agent. For simplicity in what follows we assume that D_T is bounded; this condition can be relaxed with the introduction of an appropriate Novikov-type condition; but for definiteness we will not pursue the more general situation here. Define the deterministic nonnegative process $\{\nu_t\}_{0\leq t\leq T}$ by

$$\nu_t = \sigma_t + \frac{1}{T-t}\int_0^t \sigma_s \mathrm{d}s, \tag{61}$$

and let $\{\xi_t\}$ be defined as in (55). We define the process $\{\Lambda_t\}_{0\leq t<T}$ by the relation

$$\frac{1}{\Lambda_t} = \exp\left(-D_T\int_0^t \nu_s \mathrm{d}B_s - \tfrac{1}{2}D_T^2\int_0^t \nu_s^2 \mathrm{d}s\right). \tag{62}$$

With these elements in hand, we fix a time horizon $U \in (0,T)$ and introduce a probability measure $\mathbb{B}_T$ on $\mathcal{G}_U$ by the relation

$$\mathrm{d}\mathbb{B}_T = \Lambda_U^{-1}\mathrm{d}\mathbb{Q}. \tag{63}$$

Then we have the following: (i) the process $\{W_t^*\}_{0\leq t<U}$ defined by

$$W_t^* = D_T\int_0^t \nu_s \mathrm{d}s + B_t \tag{64}$$

is a $\mathbb{B}_T$-Brownian motion; (ii) the process $\{\xi_t\}$ defined by (55) is a $\mathbb{B}_T$-Brownian bridge and is independent of D_T; (iii) the random variable D_T has the same probability law with respect to $\mathbb{B}_T$ and $\mathbb{Q}$; (iv) the conditional expectation for any integrable function $f(D_T)$ of the random variable D_T can be expressed in the form

$$\mathbb{E}^{\mathbb{Q}}[f(D_T)|\mathcal{F}_t^\xi] = \frac{\mathbb{E}^{\mathbb{B}_T}\left[f(D_T)\Lambda_t\middle|\mathcal{F}_t^\xi\right]}{\mathbb{E}^{\mathbb{B}_T}\left[\Lambda_t\middle|\mathcal{F}_t^\xi\right]}. \tag{65}$$

We note that the measure $\mathbb{B}_T$ is independent of the specific choice of the time horizon U in the sense that if $\mathbb{B}_T$ is defined on $\mathcal{G}_{U'}$ for some $U' > U$, then the restriction of that measure to $\mathcal{G}_U$ agrees with the measure $\mathbb{B}_T$ as already defined.

When we say that $\{\xi_t\}$ is a $\mathbb{B}_T$-Brownian bridge what we mean, more precisely, is that $\xi_0 = 0$, that $\mathbb{E}^{\mathbb{B}_T}[\xi_t] = 0$, and that $\mathbb{E}^{\mathbb{B}_T}[\xi_s\xi_t] = s(T-t)/T$ for all s,t such that $0 \leq s \leq t \leq U$ for any choice of the time horizon $U < T$. Thus with respect to the measure $\mathbb{B}_T$ the process $\{\xi_t\}_{0\leq t\leq U}$ has the properties of a standard $[0,T]$-Brownian bridge that has been truncated at time U. The fact that $\{\xi_t\}$ is a $\mathbb{B}_T$-Brownian bridge can be verified as follows. We have:

$$\begin{aligned}
\xi_t &= D_T\int_0^t \sigma_s \mathrm{d}s + (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}B_s \\
&= D_T\int_0^t \sigma_s \mathrm{d}s + (T-t)\int_0^t \frac{1}{T-s}\,(\mathrm{d}W_s^* - D_T\nu_s \mathrm{d}s) \\
&= D_T\left(\int_0^t \sigma_s \mathrm{d}s - (T-t)\int_0^t \frac{1}{T-s}\,\nu_s \mathrm{d}s\right) + (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}W_s^* \\
&= (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}W_s^*,
\end{aligned} \tag{66}$$

where in the final step we have made use of the relation

$$\int_0^t \frac{1}{T-s}\,\nu_s \mathrm{d}s = \frac{1}{T-t}\int_0^t \sigma_s \mathrm{d}s. \tag{67}$$

This relation can be verified explicitly by differentiation, which then gives us (61). In (66) we see that $\{\xi_t\}$ has been given the standard integral representation of a Brownian bridge. We remark, incidentally, that (65) can be thought of a variation of the Kallianpur-Striebel formula appearing in the literature of nonlinear filtering (see, for example, Bucy and Joseph [4], Davis and Marcus [7], Kallianpur and Striebel [15], Krishnan [17], and Liptser and Shiryaev [18]), the latter being applicable when β_{tT} is replaced by a standard Brownian motion.

X. DERIVATION OF THE CONDITIONAL DENSITY

We have introduced the idea of measure changes associated with Brownian bridges in order to introduce formula (65), which involves the density process $\{\Lambda_t\}$. The process $\{\Lambda_t\}$ in (62) is defined in terms of the $\mathbb{Q}$-Brownian motion $\{B_t\}$. On the other hand the expectations appearing in (65) are conditional with respect to the information generated by $\{\xi_t\}$. Therefore, it will be convenient if we express $\{\Lambda_t\}$ directly in terms of the market information process $\{\xi_t\}$. To do this we substitute (64) in (62) to obtain

$$\Lambda_t = \exp\left(D_T\int_0^t \nu_s \mathrm{d}W_s^* - \tfrac{1}{2}D_T^2\int_0^t \nu_s^2 \mathrm{d}s\right). \tag{68}$$

We then observe, by differentiating (66), that

$$\mathrm{d}\xi_t = -\frac{\xi_t}{T-t}\,\mathrm{d}t + \mathrm{d}W_t^*. \tag{69}$$

Substituting this relation in (68) we obtain

$$\Lambda_t = \exp\left[D_T\left(\int_0^t \nu_s \mathrm{d}\xi_s + \int_0^t \frac{1}{T-s}\,\nu_s\xi_s \mathrm{d}s\right) - \tfrac{1}{2}D_T^2\int_0^t \nu_s^2 \mathrm{d}s\right]. \tag{70}$$

In principle at this point all we need to do is to substitute the (61) into (70) to obtain the result for $\{\Lambda_t\}$. In practice, further simplification can be achieved. To this end, we note that by taking the differential of the coefficient of D_T in the exponent of (70) we get

$$\begin{aligned}
\mathrm{d}\left(\int_0^t \nu_s \mathrm{d}\xi_s + \int_0^t \frac{1}{T-s}\,\nu_s\xi_s \mathrm{d}s\right) &= \nu_t\left(\mathrm{d}\xi_t + \frac{1}{T-t}\,\xi_t \mathrm{d}t\right)\\
&= \left(\sigma_t + \frac{1}{T-t}\int_0^t \sigma_s \mathrm{d}s\right)\left(\mathrm{d}\xi_t + \frac{1}{T-t}\,\xi_t \mathrm{d}t\right)\\
&= \mathrm{d}\left(\frac{1}{T-t}\,\xi_t\int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right).
\end{aligned} \tag{71}$$

Then integrating both sides of (71) we obtain:

$$\int_0^t \nu_s \mathrm{d}\xi_s + \int_0^t \frac{1}{T-s}\,\nu_s\xi_s \mathrm{d}s = \frac{1}{T-t}\,\xi_t\int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s. \tag{72}$$

Similarly, by taking the differential of the coefficient of $-\frac{1}{2}D_T^2$ in the exponent of (70) and making use of (61) we find

$$\begin{aligned}\nu_t^2 \mathrm{d}t &= \left[\sigma_t^2 + 2\frac{1}{T-t}\,\sigma_t \int_0^t \sigma_s \mathrm{d}s + \frac{1}{(T-t)^2}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2\right]\mathrm{d}t \\ &= \mathrm{d}\left[\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right]. \qquad (73)\end{aligned}$$

Therefore, by integrating both sides of (73) we obtain an identity for the coefficient of $-\frac{1}{2}D_T^2$.

It follows by virtue of the two identities just obtained that the change-of-measure density process $\{\Lambda_t\}$ can be expressed in terms of the information process $\{\xi_t\}$. More explicitly,

$$\Lambda_t = \exp\left[D_T\left(\tfrac{1}{T-t}\,\xi_t \int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \tfrac{1}{2}D_T^2\left(\tfrac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)\right]. \qquad (74)$$

Note that by transforming (70) into (74) we have eliminated a term having $\{\xi_t\}$ in the integrand, thus achieving a considerable simplification. Proposition 3 can then be deduced if we use equation (67) and the basic relation

$$\mathbb{Q}\left(D_T \le x \middle| \mathcal{F}_t^\xi\right) = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{D_T \le x\}} \middle| \mathcal{F}_t^\xi\right]. \qquad (75)$$

In particular, since D_T and $\{\xi_t\}$ are independent under the bridge measure, by virtue of (67), (74), and (75) we obtain

$$\mathbb{Q}\left(D_T \le x \middle| \mathcal{F}_t^\xi\right) = \frac{\int_0^x p(y)\,\mathrm{e}^{y\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}y^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)}\mathrm{d}y}{\int_0^\infty p(y)\,\mathrm{e}^{y\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}y^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)}\mathrm{d}y}, \qquad (76)$$

from which we immediately infer Proposition 3 by differentiation.

We conclude this section by noting that an alternative expression for $\{\pi_t(x)\}$, written in terms of $\{W_t^*\}$, is given by

$$\pi_t(x) = \frac{p(x)\,\exp\left(x\int_0^t \nu_u \mathrm{d}W_u^* - \frac{1}{2}x^2\int_0^t \nu_u^2 \mathrm{d}u\right)}{\int_0^\infty p(x)\,\exp\left(x\int_0^t \nu_u \mathrm{d}W_u^* - \frac{1}{2}x^2\int_0^t \nu_u^2 \mathrm{d}u\right)\mathrm{d}x}. \qquad (77)$$

Similarly, the corresponding expression for $\{D_{tT}\}$ is given by

$$D_{tT} = \frac{\int_0^\infty x p(x)\,\exp\left(x\int_0^t \nu_u \mathrm{d}W_u^* - \frac{1}{2}x^2\int_0^t \nu_u^2 \mathrm{d}u\right)\mathrm{d}x}{\int_0^\infty p(x)\,\exp\left(x\int_0^t \nu_u \mathrm{d}W_u^* - \frac{1}{2}x^2\int_0^t \nu_u^2 \mathrm{d}u\right)\mathrm{d}x}. \qquad (78)$$

XI. DYNAMIC CONSISTENCY

Before we proceed to analyse in detail the dynamics of the price process $\{S_t\}$, first we shall establish a remarkable *dynamical consistency* condition satisfied by prices obtained in the information-based framework. By "consistency" we have in mind the following. Suppose that we re-initialise the information process at an intermediate time $s \in (0,T)$ by specifying the value ξ_s of the information at that time. For the framework to be dynamically consistent, we require that the remainder of the period $[s,T]$ admits a representation in terms of a suitably "renormalised" information process. Specifically, we have:

Proposition 4. *Let $0 \le s \le t \le T$. Then the conditional probability $\pi_t(x)$ can be written in terms of the intermediate conditional probability $\pi_s(x)$ in the form*

$$\pi_t(x) = \frac{\pi_s(x)\,\mathrm{e}^{x\left(\frac{1}{T-t}\eta_t\int_s^t\tilde{\sigma}_u\mathrm{d}u+\int_s^t\tilde{\sigma}_u\mathrm{d}\eta_u\right)-\frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_s^t\tilde{\sigma}_u\mathrm{d}u\right)^2+\int_s^t\tilde{\sigma}_u^2\mathrm{d}u\right)}}{\int_0^\infty\pi_s(x)\,\mathrm{e}^{x\left(\frac{1}{T-t}\eta_t\int_s^t\tilde{\sigma}_u\mathrm{d}u+\int_s^t\tilde{\sigma}_u\mathrm{d}\eta_u\right)-\frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_s^t\tilde{\sigma}_u\mathrm{d}u\right)^2+\int_s^t\tilde{\sigma}_u^2\mathrm{d}u\right)}\mathrm{d}x}, \tag{79}$$

where

$$\tilde{\sigma}_u = \sigma_u + \frac{1}{T-s}\int_0^s \sigma_v\mathrm{d}v \tag{80}$$

is the re-initialised market information flow rate, and

$$\eta_t = \xi_t - \frac{T-t}{T-s}\,\xi_s \tag{81}$$

is the re-initialised information process.

The fact that $\{\eta_t\}_{s\le t\le T}$ represents the updated information process bridging the interval $[s,T]$ can be seen as follows. First we note that $\eta_s = 0$ and that $\eta_T = \xi_T$. Substituting (55) in (81) we find that

$$\eta_t = D_T\int_s^t \tilde{\sigma}_u\mathrm{d}u + \gamma_{tT}, \tag{82}$$

where $\tilde{\sigma}_u$ is as defined in (80), and

$$\gamma_{tT} = \beta_{tT} - \frac{T-t}{T-s}\,\beta_{sT}. \tag{83}$$

A short calculation making use of the covariance of the Brownian bridge $\{\beta_{tT}\}$ shows that the Gaussian process $\{\gamma_{tT}\}_{s\le t\le T}$ is a standard Brownian bridge over the interval $[s,T]$. It thus follows that $\{\eta_t\}$ is the information bridge interpolating the interval $[s,T]$.

To verify (79) we note that (77) can be written as

$$\pi_t(x) = \frac{\pi_s(x)\,\exp\left(x\int_s^t\nu_u\mathrm{d}W_u^* - \frac{1}{2}x^2\int_s^t\nu_u^2\mathrm{d}u\right)}{\int_0^\infty\pi_s(x)\,\exp\left(x\int_s^t\nu_u\mathrm{d}W_u^* - \frac{1}{2}x^2\int_s^t\nu_u^2\mathrm{d}u\right)\mathrm{d}x}. \tag{84}$$

The identity given in (71) then implies that

$$\begin{aligned}\int_s^t \nu_u\mathrm{d}W_u^* &= \frac{1}{T-t}\,\xi_t\int_s^t\sigma_u\mathrm{d}u + \int_s^t\sigma_u\mathrm{d}\xi_u + \left(\frac{\xi_t}{T-t} - \frac{\xi_s}{T-s}\right)\int_0^s\sigma_u\mathrm{d}u \\ &= \frac{1}{T-t}\,\eta_t\int_s^t\tilde{\sigma}_u\mathrm{d}u + \int_s^t\tilde{\sigma}_u\mathrm{d}\eta_u, \end{aligned} \tag{85}$$

where we have made use of (80) and (81). Similarly, the relation in (71) implies

$$\begin{aligned}\int_s^t \nu_u^2 \mathrm{d}u &= \frac{1}{T-t}\left(\int_0^t \sigma_u \mathrm{d}u\right)^2 - \frac{1}{T-s}\left(\int_0^s \sigma_u \mathrm{d}u\right)^2 + \int_s^t \sigma_u^2 \mathrm{d}u \\ &= \frac{1}{T-t}\left(\int_s^t \tilde{\sigma}_u \mathrm{d}u\right)^2 + \int_s^t \tilde{\sigma}_u^2 \mathrm{d}u. \end{aligned} \tag{86}$$

Substitution of (85) and (86) into (84) establishes (79). In particular, the form of (79) is identical to the original formula (58), modulo the indicated renormalisation of the information process and the associated information flow rate.

XII. EXPECTED DIVIDEND PROCESS

The goal of sections VIII, IX, and X was to obtain an expression for the conditional expectation (13) in the case of a single-dividend asset in the situation of a time-dependent information flow rate. In the analysis of the associated price process it will therefore be useful to work out the dynamics of the conditional expectation of the dividend. In particular, an application of Ito's rule to (59), after some rearrangement of terms, shows that

$$\mathrm{d}D_{tT} = \nu_t V_t \left(\frac{1}{T-t}\,\xi_t - \nu_t D_{tT}\right)\mathrm{d}t + \nu_t V_t \mathrm{d}\xi_t, \tag{87}$$

where $\{V_t\}$ is the conditional variance of the random variable D_T:

$$V_t = \int_0^\infty x^2 \pi_t(x)\mathrm{d}x - \left(\int_0^\infty x\pi_t(x)\mathrm{d}x\right)^2. \tag{88}$$

Let us define a new process $\{W_t\}$ according to the prescription

$$W_t = \xi_t + \int_0^t \frac{1}{T-s}\,\xi_s \mathrm{d}s - \int_0^t \nu_s D_{sT}\mathrm{d}s. \tag{89}$$

We refer to $\{W_t\}$ as the "innovation process". It follows from the definition of $\{W_t\}$ that

$$\mathrm{d}D_{tT} = \nu_t V_t \, \mathrm{d}W_t. \tag{90}$$

Since $\{D_{tT}\}$ is an $\{\mathcal{F}_t\}$-martingale we are thus led to conjecture that $\{W_t\}$ must also be an $\{\mathcal{F}_t\}$-martingale. In fact, we have the following result:

Proposition 5. *The process $\{W_t\}$ defined by* (89) *is a standard $\{\mathcal{F}_t\}$-Brownian motion under the risk-neutral measure $\mathbb{Q}$.*

Proof. To show this we shall establish that (i) $\{W_t\}$ is an $\{\mathcal{F}_t^\xi\}$-martingale, and that (ii) $(\mathrm{d}W_t)^2 = \mathrm{d}t$. Writing as before $\mathbb{E}_t^{\mathbb{Q}}[-] = \mathbb{E}^{\mathbb{Q}}[-|\mathcal{F}_t^\xi]$ for the conditional expectation and letting $t \leq u$ we have

$$\mathbb{E}_t^{\mathbb{Q}}\left[W_u\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\xi_u\right] + \mathbb{E}_t^{\mathbb{Q}}\left[\int_0^u \frac{1}{T-s}\,\xi_s \mathrm{d}s\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\int_0^u \nu_s D_{sT}\mathrm{d}s\right]. \tag{91}$$

Splitting the second two terms on the right into integrals between 0 and t, and between t and u, we thus obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_t\left[W_u\right] \;=\; & \mathbb{E}^{\mathbb{Q}}_t[\xi_u]+\int_0^t \frac{1}{T-s}\,\xi_s\mathrm{d}s-\int_0^t \nu_s D_{sT}\mathrm{d}s \\ & +\int_t^u \frac{1}{T-s}\,\mathbb{E}^{\mathbb{Q}}_t[\xi_s]\mathrm{d}s-\int_t^u \nu_s\mathbb{E}^{\mathbb{Q}}_t[D_{sT}]\mathrm{d}s. \end{aligned} \tag{92}$$

The martingale property of the conditional expectation implies that $\mathbb{E}^{\mathbb{Q}}_t[D_{sT}] = D_{tT}$ for $t \leq s$, which allows us to simplify the last term. To simplify the expression for the conditional expectation $\mathbb{E}^{\mathbb{Q}}_t[\xi_s]$ for $t \leq s$ we use the tower property of conditional expectation:

$$\mathbb{E}^{\mathbb{Q}}_t[\beta_{sT}] = \mathbb{E}^{\mathbb{Q}}_t[\mathbb{E}[\beta_{sT}|H_T,\beta_{tT}]] = \mathbb{E}^{\mathbb{Q}}_t[\mathbb{E}[\beta_{sT}|\beta_{tT}]]. \tag{93}$$

for $t \leq s$. To calculate the inner expectation $\mathbb{E}[\beta_{sT}|\beta_{tT}]$ here we use the fact that the random variable $\beta_{sT}/(T-s) - \beta_{tT}(T-t)$ is independent of β_{tT} and deduce that

$$\mathbb{E}[\beta_{sT}|\beta_{tT}] = \frac{T-s}{T-t}\,\beta_{tT}, \tag{94}$$

from which it follows that

$$\mathbb{E}^{\mathbb{Q}}_t\left[\beta_{sT}\right] = \frac{T-s}{T-t}\,\mathbb{E}^{\mathbb{Q}}_t[\beta_{tT}]. \tag{95}$$

As a result we obtain

$$\mathbb{E}^{\mathbb{Q}}_t[\xi_s] = D_{tT}\int_0^s \sigma_v\mathrm{d}v + \frac{T-s}{T-t}\,\mathbb{E}^{\mathbb{Q}}_t[\beta_{tT}]. \tag{96}$$

We also recall the definition of $\{W_t\}$ given by equation (89), which implies that

$$\int_0^t \frac{1}{T-s}\,\xi_s\mathrm{d}s - \int_0^t \nu_s D_{sT}\mathrm{d}s = W_t - \xi_t. \tag{97}$$

Therefore, substituting (96) and (97) into (92) we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_t\left[W_u\right] \;=\; & D_{tT}\int_0^u \sigma_s\mathrm{d}s + W_t - \xi_t + D_{tT}\int_t^u \frac{1}{T-s}\Big(\int_0^s \sigma_v\mathrm{d}v\Big)\mathrm{d}s - D_{tT}\int_t^u \nu_s\mathrm{d}s \\ & +\mathbb{E}^{\mathbb{Q}}_t[\beta_{tT}]. \end{aligned} \tag{98}$$

Next we split the first term into an integral from 0 to t and an integral from t to u, and we insert the definition (61) of $\{\nu_t\}$ into the fifth term. The result is:

$$\mathbb{E}^{\mathbb{Q}}_t\left[W_u\right] \;=\; W_t + D_{tT}\int_0^t \sigma_s\mathrm{d}s + \mathbb{E}^{\mathbb{Q}}_t[\beta_{tT}] - \xi_t. \tag{99}$$

Finally, if we make use of the fact that $\xi_t = \mathbb{E}^{\mathbb{Q}}_t[\xi_t]$, and hence that

$$\xi_t = D_{tT}\int_0^t \sigma_s\mathrm{d}s + \mathbb{E}^{Q}_t[\beta_{tT}], \tag{100}$$

it follows that $\{W_t\}$ satisfies the martingale condition. On the other hand, by virtue of (89) we have $(\mathrm{d}W_t)^2 = \mathrm{d}t$. We thus conclude that $\{W_t\}$ is an $\{\mathcal{F}^{\xi}_t\}$-Brownian motion. $\square$

XIII. ASSET PRICES AND DERIVATIVE PRICES

We are now in a position to consider in more detail the dynamics of the price process of an asset paying a single dividend D_T in the case of a time-dependent information flow. For $\{S_t\}$ we have $S_t = \mathbf{1}_{\{t<T\}} P_{tT} D_{tT}$, or equivalently

$$S_t = \mathbf{1}_{\{t<T\}} P_{tT} \frac{\int_0^\infty x p(x)\, \mathrm{e}^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)} \mathrm{d}x}{\int_0^\infty p(x)\, \mathrm{e}^{x\left(\frac{1}{T-t}\xi_t \int_0^t \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\xi_s\right) - \frac{1}{2}x^2\left(\frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s\right)} \mathrm{d}x}. \tag{101}$$

A straightforward calculation making use of (90) shows that for the dynamics of the price process we have

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \Gamma_{tT} \mathrm{d}W_t, \tag{102}$$

where the asset price volatility process $\{\Gamma_{tT}\}$ is given by

$$\Gamma_{tT} = \nu_t P_{tT} V_t., \tag{103}$$

where V_t is the conditional variance of the dividend:

$$V_t = \mathbb{E}_t^{\mathbb{Q}}\left[\left(D_T - \mathbb{E}_t^{\mathbb{Q}}[D_T]\right)^2\right]. \tag{104}$$

It should be evident by virtue of its definition that $\{V_t\}$ is a supermartingale. More specifically, for the dynamics of $\{V_t\}$ we obtain

$$\mathrm{d}V_t = -\nu_t^2 V_t^2 \mathrm{d}t + \nu_t \kappa_t \mathrm{d}W_t, \tag{105}$$

where κ_t denotes the third conditional moment of D_T, given by

$$\kappa_t = \mathbb{E}_t^{\mathbb{Q}}\left[(D_T - D_{tT})^3\right]. \tag{106}$$

Although we have derived formula (101) by assuming that the price process is induced by the market information $\{\xi_t\}$, the result to be shown in Section XIV below demonstrates that we can regard the dynamical equation (102) for the price process as given, and then deduce the structure of the underlying information process. The information-based interpretation of the modelling framework, however, is more appealing. According to this interpretation there is a flow of market information, which is available to all market participants and is represented by the filtration generated by the information process $\{\xi_t\}$. Given this information, each participant will "act", in our interpretation, so as to minimise the risk adjusted future P&L variance associated with the cash flow under consideration. The future P&L is determined by the value of the random variable D_T, and the estimate of D_T that minimises its variance is indeed given by the conditional expectation (13). By discounting this expectation with P_{tT} we recover the induced price process $\{S_t\}$.

As for the volatility of the asset price, we note that $\{\Gamma_{tT}\}$ is "infinitely stochastic" in the sense that all the higher-order volatilities (the volatility of the volatility, and so on) are stochastic. Furthermore, these higher-order volatilities have a natural interpretation: the volatility of the asset price is determined by the variance of the random cash flow; the volatility of the volatility is determined by the skewness of D_T; its volatility is determined by the kurtosis of D_T; and so on.

The fact that the asset price in the transformed probability measure is given by a function of a Gaussian random variable means that the pricing of various derivatives is numerically straightforward. We have seen this already in the case of a constant information-flow rate, but the result remains valid in the time-dependent case as well. For example, consider a European-style call option on the asset with strike K and maturity t, where $t \leq T$, for which the price is given as in (23). Then by changing the measure we obtain the simple formula

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left\{\int_0^\infty (P_{tT}x - K)p(x)\,\exp\left(x\int_0^t \nu_s \mathrm{d}W_s^* - \tfrac{1}{2}x^2\int_0^t \nu_s^2 \mathrm{d}s\right)\mathrm{d}x\right\}^+\right] \quad (107)$$

for the value of the option. This result should be compared with equation (27). We note that in the bridge measure the expression $\int_0^t \nu_s \mathrm{d}W_s^*$ is a Gaussian random variable with mean zero and variance

$$\omega_t^2 = \int_0^t \nu_s^2 \mathrm{d}s, \quad (108)$$

where

$$\int_0^t \nu_s^2 \mathrm{d}s = \frac{1}{T-t}\left(\int_0^t \sigma_s \mathrm{d}s\right)^2 + \int_0^t \sigma_s^2 \mathrm{d}s. \quad (109)$$

Here we have used the relation established in (73). Therefore, if we set

$$Y = \omega_t^{-1}\int_0^t \nu_s \mathrm{d}W_s^*, \quad (110)$$

it follows immediately that Y is a standard normal random variable in the bridge measure. For the call price we thus have

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{B}_T}\left[\left\{\int_0^\infty (P_{tT}x - K)p(x)\mathrm{e}^{\omega_t x Y - \frac{1}{2}\omega_t^2 x^2}\mathrm{d}x\right\}^+\right], \quad (111)$$

and hence

$$C_0 = P_{0t}\frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^\infty \mathrm{e}^{-\frac{1}{2}y^2}\left(\int_{x=0}^\infty (P_{tT}x - K)p(x)\mathrm{e}^{\omega_t x y - \frac{1}{2}\omega_t^2 x^2}\mathrm{d}x\right)^+ \mathrm{d}y. \quad (112)$$

We observe that there exists a critical value $y = y^*$ such that the argument of the "positive-part-of" function vanishes in the expression above. Thus y^* is given by

$$\int_0^\infty (P_{tT}x - K)p(x)\mathrm{e}^{\omega_t x y^* - \frac{1}{2}\omega_t^2 x^2}\mathrm{d}x = 0. \quad (113)$$

As a consequence the call price can be written in the form

$$C_0 = P_{0t}\frac{1}{\sqrt{2\pi}}\int_{y=y^*}^\infty \mathrm{e}^{-\frac{1}{2}y^2}\left(\int_{x=0}^\infty (P_{tT}x - K)p(x)\mathrm{e}^{\omega_t x y - \frac{1}{2}\omega_t^2 x^2}\mathrm{d}x\right)\mathrm{d}y. \quad (114)$$

The integration in the y variable can be performed, and we deduce the following elementary representation for the call price:

$$C_0 = P_{0t}\int_0^\infty (P_{tT}x - K)p(x)N(\omega_t x - y^*)\mathrm{d}x. \quad (115)$$

When the cash flow is represented by a discrete random variable and the information-flow rate is constant, this result reduces to an expression equivalent to the option pricing formula derived in Brody, *et al.* [3]. More generally, if the cash flow is a continuous random variable and the information flow rate is constant then we recover the expression (31) given in section IV (see also Rutkowski and Yu [20]).

We conclude this section with the remark that the simulation of the price process $\{S_t\}$ is straightforward in the present scheme. First, we generate a Brownian trajectory $\{\gamma_t(\omega)\}$, and form the associated Brownian bridge trajectory $\{\beta_{tT}(\omega)\}$. We then select a value for D_T by a method consistent with the *a priori* probability density $p(x)$, and substitute these in the formula $\xi_t(\omega) = D_T(\omega)\int_0^t \sigma_s \mathrm{d}s + \beta_{tT}(\omega)$ for some appropriate choice of $\{\sigma_t\}$. Finally, substitution of $\{\xi_t(\omega)\}$ in (101) gives us a simulated path $\{S_t(\omega)\}$. The statistics of the price process $\{S_t\}$ in the risk-neutral measure are then readily obtained by repeating this procedure, the results of which can be used to price derivatives, or to calibrate the time-dependent information-flow rate $\{\sigma_t\}$.

XIV. EXISTENCE OF THE INFORMATION PROCESS

In this section we consider what might appropriately be called the "inverse" problem for information-based asset pricing. In the inverse problem the idea is to begin with the conditional probability density process $\{\pi_t(x)\}$ and to construct from it the independent degrees of freedom represented by the X-factor D_T and the noise $\{\beta_{tT}\}$. The setup can be described more specifically as follows. On the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ let $\{W_t\}$ be a Brownian motion and let $\{\mathcal{F}_t\}$ be the filtration generated by $\{W_t\}$. Let D_T be an $\mathcal{F}_T$-measurable random variable, and let $\{\pi_t(x)\}$ denote the associated conditional probability density process. We assume that $\{\pi_t(x)\}$ satisfies the stochastic differential equation

$$\mathrm{d}\pi_t(x) = \nu_t(x - D_{tT})\pi_t(x)\,\mathrm{d}W_t, \tag{116}$$

with a prescribed initial condition $\pi_0(x) = p(x)$, where $\{\nu_t\}$ is given by (61), and D_{tT} is the conditional expectation

$$D_{tT} = \int_0^\infty x\pi_t(x)\,\mathrm{d}x. \tag{117}$$

We define the process $\{\xi_t\}$ as follows:

$$\xi_t = (T-t)\int_0^t \frac{1}{T-s}\Big(\mathrm{d}W_s + \nu_s D_{sT}\mathrm{d}s\Big). \tag{118}$$

Then we have the following result:

Proposition 6. *The random variables D_T and $\beta_{tT} = \xi_t - D_T\int_0^t \sigma_s \mathrm{d}s$ are $\mathbb{Q}$-independent for all $t \in [0, T]$. Furthermore, the process $\{\beta_{tT}\}$ is a $\mathbb{Q}$-Brownian bridge.*

Proof. To establish the independence of D_T and β_{tT} it suffices to verify that

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{x\beta_{tT}+yD_T}] = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{x\beta_{tT}}]\,\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{yD_T}] \tag{119}$$

for arbitrary x, y. Using the tower property of conditional expectation we have

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{x\beta_{tT}+yD_T}] = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\xi_t}\,\mathbb{E}^{\mathbb{Q}}_t\left[\mathrm{e}^{(y-x\int_0^t \sigma_s \mathrm{d}s)D_T}\right]\right], \tag{120}$$

where we have inserted the definition of the β_{tT} given in the statement of the Proposition. We consider the inner expectation first. From equation (65) for the conditional expectation of a function of the random variable D_T we deduce that

$$\mathbb{E}^{\mathbb{Q}}_t\left[\mathrm{e}^{(y-x\int_0^t\sigma_s\mathrm{d}s)D_T}\right]=\Phi_t^{-1}\int_0^\infty p(z)\,\mathrm{e}^{(y-x\int_0^t\sigma_s\mathrm{d}s)z}\,\mathrm{e}^{z\int_0^t\nu_u\mathrm{d}W^*_u-\frac{1}{2}z^2\int_0^t\nu_u^2\mathrm{d}u}\mathrm{d}z, \tag{121}$$

where the process $\{\Phi_t\}$ is defined by

$$\Phi_t=\int_0^\infty p(z)\exp\left(z\int_0^t\nu_u\mathrm{d}W^*_u-\tfrac{1}{2}z^2\int_0^t\nu_u^2\mathrm{d}u\right)\mathrm{d}z. \tag{122}$$

In other words,

$$\Phi_t=\mathbb{E}^{\mathbb{Q}}[\Lambda_t|\mathcal{F}^\xi_t]. \tag{123}$$

We now change the probability measure from $\mathbb{Q}$ to $\mathbb{B}_T$, so that the term Φ_t^{-1} appearing in (121) drops out to give us

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\xi_t}\,\mathbb{E}^{\mathbb{Q}}_t\left[\mathrm{e}^{(y-x\int_0^t\sigma_s\mathrm{d}s)D_T}\right]\right]\\
&\quad=\int_0^\infty p(z)\,\mathbb{E}^{\mathbb{B}_T}\left[\mathrm{e}^{x(T-t)\int_0^t\frac{1}{T-s}\mathrm{d}W^*_s+(y-x\int_0^t\sigma_s\mathrm{d}s)z+z\int_0^t\nu_s\mathrm{d}W^*_s-\frac{1}{2}z^2\int_0^t\nu_s^2\mathrm{d}s}\right]\mathrm{d}z\\
&\quad=\int_0^\infty p(z)\,\mathrm{e}^{(y-x\int_0^t\sigma_s\mathrm{d}s)z-\frac{1}{2}z^2\int_0^t\nu_s^2\mathrm{d}s+\frac{1}{2}\int_0^t\alpha_s^2\mathrm{d}s}\,\mathbb{E}^{\mathbb{B}_T}\left[\mathrm{e}^{\int_0^t\alpha_s\mathrm{d}W^*_s-\frac{1}{2}\int_0^t\alpha_s^2\mathrm{d}s}\right]\mathrm{d}z,
\end{aligned} \tag{124}$$

where $\alpha_s=x(T-t)/(T-s)+z\nu_s$, and therefore

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{x\beta_{tT}+yD_T}]=\int_0^\infty p(z)\,\mathrm{e}^{(y-x\int_0^t\sigma_s\mathrm{d}s)z-\frac{1}{2}z^2\int_0^t\nu_s^2\mathrm{d}s+\frac{1}{2}\int_0^t\alpha_s^2\mathrm{d}s}\mathrm{d}z \tag{125}$$

Furthermore, making use of relation (67) we have

$$\exp\left(-xz\int_0^t\sigma_s\mathrm{d}s-\frac{1}{2}z^2\int_0^t\nu_s^2\mathrm{d}s+\frac{1}{2}\int_0^t\alpha_s^2\mathrm{d}s\right)=\exp\left(\frac{t(T-t)}{2T}\,x^2\right). \tag{126}$$

As a consequence, we immediately infer from (125) that

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\beta_{tT}+yD_T}\right]=\left(\int_0^\infty p(z)\,\mathrm{e}^{yz}\mathrm{d}z\right)\exp\left(\frac{t(T-t)}{2T}x^2\right), \tag{127}$$

and thus factorises into the product of a function of x and a function of y. This establishes the independence of $\{\beta_{tT}\}$ and D_T.

Equation (127) also shows that the process $\{\beta_{tT}\}$ is $\mathbb{Q}$-Gaussian, with mean zero and variance $t(T-t)/T$. To establish that $\{\beta_{tT}\}$ is a Brownian bridge, we must show that for $s\le t$ the covariance of β_{sT} and β_{tT} is given by $s(T-t)/T$. Alternatively, it suffices to analyse the moment generating function $\mathbb{E}[\mathrm{e}^{x\beta_{sT}+y\beta_{tT}}]$. We proceed as follows. First, using the tower property we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\beta_{sT}+y\beta_{tT}}\right]&=\mathbb{E}\left[\mathrm{e}^{x\xi_s+y\xi_t-(x\int_0^s\sigma_u\mathrm{d}u+y\int_0^t\sigma_u\mathrm{d}u)D_T}\right]\\
&=\mathbb{E}\left[\mathrm{e}^{x\xi_s+y\xi_t}\mathbb{E}^{\mathbb{Q}}_t\left[\mathrm{e}^{-(x\int_0^s\sigma_u\mathrm{d}u+y\int_0^t\sigma_u\mathrm{d}u)D_T}\right]\right].
\end{aligned} \tag{128}$$

Next, by use of formula (65), the inner expectation can be carried out to give

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\beta_s+y\beta_{tT}}\right]=\mathbb{E}\left[\mathrm{e}^{x\xi_s+y\xi_t}\Phi_t^{-1}\int_0^\infty p(z)\,\mathrm{e}^{-(x\int_0^s\sigma_u\mathrm{d}u+y\int_0^t\sigma_u\mathrm{d}u)z}\,\mathrm{e}^{z\int_0^t\nu_u\mathrm{d}W_u^*-\frac{1}{2}z^2\int_0^t\nu_u^2\mathrm{d}u}\mathrm{d}z\right] \tag{129}$$

If we change the probability measure to $\mathbb{B}_T$ the process $\{\Phi_t\}$ in the denominator drops out, and we have

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\beta_s+y\beta_{tT}}\right]=\int_0^\infty p(z)\,\mathrm{e}^{-(x\int_0^s\sigma_u\mathrm{d}u+y\int_0^t\sigma_u\mathrm{d}u)z-\frac{1}{2}z^2\int_0^t\nu_u^2\mathrm{d}u}\,\mathbb{E}^{\mathbb{B}_T}\left[\mathrm{e}^{x\xi_s+y\xi_t+z\int_0^t\nu_s\mathrm{d}W_s^*}\right]\mathrm{d}z. \tag{130}$$

Let us consider the inner expectation first. By defining $a_u=x(T-s)/(T-u)$ and $b_u=y(T-t)/(T-u)+z\nu_u$ we can write

$$\mathbb{E}^{\mathbb{B}_T}\left[\mathrm{e}^{x\xi_s+y\xi_t+z\int_0^t\nu_s\mathrm{d}W_s^*}\right]=\mathbb{E}^{\mathbb{B}_T}\left[\mathrm{e}^{\int_0^s a_u\mathrm{d}W_u^*+\int_0^t b_u\mathrm{d}W_u^*}\right]. \tag{131}$$

However, since $\{W_t^*\}$ is a $\mathbb{B}_T$-Brownian motion, using the properties of Gaussian random variable, we find that

$$\mathbb{E}^{\mathbb{B}_T}\left[\mathrm{e}^{\int_0^s a_u\mathrm{d}W_u^*+\int_0^t b_u\mathrm{d}W_u^*}\right]=\exp\left\{\tfrac{1}{2}\left(\int_0^s a_u^2\mathrm{d}u+\int_0^t b_u^2\mathrm{d}u+2\int_0^s a_ub_u\mathrm{d}u\right)\right\}. \tag{132}$$

Substituting the definitions for $\{a_u\}$ and $\{b_u\}$ into the right-hand side of (132) and combining the result with the remaining terms in the exponent of the right-hand side of (130) we find that the terms involving the integration variable z drop out, and we are left with the integral of the density function $p(z)$, which is of course unity. Gathering the remaining terms we then obtain

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\beta_{sT}+y\beta_{tT}}\right]=\exp\left\{\tfrac{1}{2}\left(x^2\,\frac{s(T-s)}{T}+y^2\,\frac{t(T-t)}{T}+2xy\,\frac{s(T-t)}{T}\right)\right\}. \tag{133}$$

It follows that the covariance of β_{sT} and β_{tT} for $s\leq t$ is given by

$$\frac{\partial^2}{\partial x\partial y}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{x\beta_{sT}+y\beta_{tT}}\right]\Bigg|_{x=y=0}=\frac{s(T-t)}{T}. \tag{134}$$

This establishes the assertion that $\{\beta_{tT}\}$ is a $\mathbb{Q}$-Brownian bridge. □

The result above shows that, for the class of price processes we are considering, even if at the outset we take the "usual" point of view in financial modelling, and regard the price process of the asset as being adapted to some "prespecified" filtration, nevertheless it is possible to *deduce* the structure of the underlying information-based model.

XV. MULTI-FACTOR MODELS WITH A TIME-DEPENDENT INFORMATION FLOW RATE

Let us now turn to consider the case of a single cash flow D_T that depends on a *multiplicity* of market factors $\{X^{\alpha}_{T_k}\}^{\alpha=1,\ldots,m_k}_{k=1,\ldots,n}$, where we have the n pre-designated information dates $\{T_k\}_{k=1,2,\ldots,n}$, and where for each value of k we have a set of m_k market factors. For simplicity we set $T = T_n$. Each market factor $X^\alpha_{T_k}$ is associated with a corresponding information process defined by

$$\xi^\alpha_{tT_k} = X^\alpha_{T_k} \int_0^t \sigma^\alpha_{sT_k} \mathrm{d}s + \beta^\alpha_{tT_k}, \tag{135}$$

where $X^\alpha_{T_k}$ and $\beta^\alpha_{tT_k}$ are independent. It should be evident that although the random variable D_T representing the cash flow is $\mathcal{F}_T$-measurable, the values of some of the X-factors upon which it depends may be revealed at earlier times. That is to say, the uncertainties arising from some of the economic elements affecting the value of the cash flow at time T may vanish before that time. One of the advantages of the present modelling framework is the fact that we are able to accommodate such complicated structures in a tractable theory.

Since the X-factors are independent, it follows that for each market factor the associated conditional density process $\pi^\alpha_{tT_k}(x)$ takes the form given in equation (58), and the corresponding dynamical evolution is given by

$$\mathrm{d}\pi^\alpha_{tT_k} = \nu^\alpha_{tT_k} \left(x^\alpha_k - \mathbb{E}^{\mathbb{Q}} \left[X^\alpha_{T_k} | \mathcal{F}_t \right] \right) \pi^\alpha_{tT_k} \, \mathrm{d}W^{\alpha k}_t. \tag{136}$$

The function $\nu^\alpha_{tT_k}$ appearing in this equation is given by an expression of the form (61):

$$\nu^\alpha_{tT_k} = \sigma^\alpha_{tT_k} + \frac{1}{T_k - t} \int_0^t \sigma^\alpha_{sT_k} ds. \tag{137}$$

and the innovation process $\{W^{\alpha k}_t\}$ is defined in terms of $\{\xi^\alpha_{tT_k}\}$ via a relation of the form

$$W^{\alpha k}_t = \xi^\alpha_{tT_k} + \int_0^t \frac{1}{T_k - s} \xi^\alpha_{sT_k} ds - \int_0^t \nu^\alpha_{sT_k} X^\alpha_{T_k} ds. \tag{138}$$

The conditional expectation $\mathbb{E}^{\mathbb{Q}}[D_T|\mathcal{F}_t]$ is thus given by the multi-dimensional integral

$$\begin{aligned} D_{tT} \; = \; & \int_0^\infty \cdots \int_0^\infty \Delta_T(x^1_1, \ldots, x^{m_1}_1, \ldots, x^1_n, \ldots, x^{m_n}_n) \\ & \times \pi_{t1}(x^1_1) \cdots \pi_{t1}(x^{m_1}_1) \cdots \pi_{tn}(x^1_n) \cdots \pi_{tn}(x^{m_n}_n) \, \mathrm{d}x^1_1 \cdots \mathrm{d}x^{m_1}_1 \cdots \mathrm{d}x^1_n \cdots \mathrm{d}x^{m_n}_n. \end{aligned} \tag{139}$$

The price of the asset for $t < T$ is $S_t = P_{tT} D_{tT}$. A straightforward application of Ito's rule then establishes the following result:

Proposition 7. *The price process $\{S_t\}$ of an asset that pays a single dividend D_T at time $T(= T_n)$ depending on the market factors $\{X^\alpha_{T_k}\}^{\alpha=1,2,\ldots,m_k}_{k=1,2,\ldots n}$, satisfies the dynamical equation*

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sum_{k=1}^{n} \sum_{\alpha=1}^{m_k} \nu^\alpha_{tT_k} \mathrm{Cov}_t[\Delta_T, X^\alpha_{T_k}] \, \mathrm{d}W^{\alpha k}_t, \tag{140}$$

where

$$\Delta_T = \Delta_T \left(X^\alpha_{T_1}, \ldots, X^\alpha_{T_k} \right). \tag{141}$$

Here $\mathrm{Cov}_t[\Delta_T, X^\alpha_{T_k}]$ denotes the covariance between the cash-flow function Δ_T and the market factor $X^\alpha_{T_k}$, conditional on the information $\mathcal{F}_t$ generated by the market information processes $\{\xi^\alpha_{tT_k}\}$ up to time t.

In the more general case of an asset that pays multiple dividends (see Section VI) the price process is given by

$$S_t = \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[\Delta_{T_k} \left(\{X^{\alpha}_{T_j}\}^{\alpha=1,2,\dots,m_j}_{j=1,\dots,k} \right) \Big| \, \mathcal{F}_t \right]. \tag{142}$$

Proposition 8. *The price process $\{S_t\}$ of an asset that pays the random dividends D_{T_k} on the cash flow dates T_k $(k = 1, \dots, n)$ satisfies the dynamical equation*

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sum_{k=1}^{n} \sum_{\alpha=1}^{m_k} \mathbf{1}_{\{t<T_k\}} \nu^{\alpha}_{tT_k} \mathrm{Cov}_t[\Delta_{T_k}, X^{\alpha}_{T_k}] \, \mathrm{d}W^{\alpha k}_t + \Delta_{T_k} \mathrm{d}\mathbf{1}_{\{t<T\}}, \tag{143}$$

where

$$\Delta_{T_k} = \Delta_{T_k} \left(\{X^{\alpha}_{T_j}\}^{\alpha=1,\dots,m_j}_{j=1,\dots,k} \right). \tag{144}$$

Here $\mathrm{Cov}_t[\Delta_{T_k}, X^{\alpha}_{T_k}]$ denotes the covariance between the dividend structure Δ_{T_k} and the market factor $X^{\alpha}_{T_k}$, conditional on the market information $\mathcal{F}_t$.

We conclude that the multi-factor, multi-dividend situation is also fully tractable when the information-flow rates associated with the various market factors are time dependent. A straightforward extension of Proposition 8 then allows us to formulate the joint price dynamics of a system of assets, the associated dividend flows of which may depend on common market factors. As a consequence, it follows that a rather specific model for stochastic volatility and correlation emerges for such a system of assets, and it is one of the principle conclusions of this paper that such a model, which is entirely natural in character, can indeed be formulated. The information-based "X-factor" approach presented here thus offers a fundamental new insight into the nature of volatility and correlation, and as such may find applications in a number of different areas of financial risk analysis. We have in mind, in particular, applications to equity portfolios, credit portfolios, and insurance, all of which exhibit important intertemporal market correlation effects. We also have in mind the problem of firm-wide risk management and optimal capital allocation for banking institutions.

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